Partial and random covering times in one dimension

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We study the recently introduced random walk problems of partial covering time (PCT) and random covering time (RCT). We generalize the concept of first-passage time to a given set of m sites by considering the probability of visiting all m sites for the first time on the tth step. For the one-dimensional case we derive an explicit result for the mean time needed to visit m sites for the first time. Using this result we are able to solve the PCT and RCT problems exactly in one dimension.

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I. INTRODUCTION

The random walk problem has enjoyed continual interest since the beginning of the century due to its relevance to a wide range of applications [1,2]. About a decade ago, one of the authors and collaborators [3,4] investigated the lattice covering time problem on d-dimensional hypercubic lattices using Monte Carlo (MC) simulations. The covering time (CT) t_c is the mean time taken by a walker to visit all N sites of a lattice. A theory of the d-dimensional lattice CT based on a formalism due to Rubin and Weiss [5] has been presented [4]. However, the algebra becomes quite involved as N grows and an exact derivation of t_c from this approach appears very difficult. In one dimension the CT problem can be related to the first-passage time problem [6]. For periodic boundary conditions the exact result is given by

$$t_c=\frac{1}{2}N(N-1) \quad (d=1), \tag{1.1}$$

a result supported by direct enumeration and MC simulations [3]. In two and higher dimensions, it has been conjectured [3] on the basis of MC simulations that, for $N \gg 1$,

$$t_c \approx A_2 N \ln^2 N\left(1+\frac{C_2}{\ln N}\right) \quad (d=2), \tag{1.2}$$

$$t_c \approx A_d N \ln N\left(1+\frac{C_d}{\ln N}\right) \quad (d \geqslant 3), \tag{1.3}$$

where A_d depends on dimensionality and lattice type but not on the boundary conditions, whereas C_d depends on all these factors. The asymptotic behavior of t_c for $d \geqslant 2$ was confirmed in Ref. [7] by identifying it with the characteristic time to visit the last site in the context of the problem of calculating the average number of distinct sites visited by the random walker at time t for large N. For hypercubic lattices with periodic boundary conditions, it was found [7] that $A_2 = 1/\pi = 0.318\ldots$, $A_3 = 1.516\ldots$, $A_4 = 1.239\ldots$ in very good agreement with MC results. Also for the mean field ($d=\infty$) case it has been shown that $A_\infty = 1$ [4,7].

The CT problem was generalized by considering a subset of $m \leqslant N$ sites of the lattice [8]. In the first generalization, called the partial covering time (PCT) problem, the PCT t_p is the mean time taken by a random walker to visit m distinct sites. In the second generalization, the random covering time (RCT) problem, the RCT t_r is the mean time taken by the walker to visit m sites chosen at random. Based on a detailed analysis of the MC data in $d=2$ [8], it has been suggested that, in the thermodynamic limit, the following complementarity relation between the RCT and the PCT holds:

$$\left(\frac{t_r}{t_c}\right)_f \leftrightarrow 1-\left(\frac{t_p}{t_c}\right)_{1-f}; \tag{1.4}$$

i.e., the reduced time needed to visit $m=fN$ sites previously chosen at random is equal to 1 minus the reduced time needed to visit the complementary number $N-m=(1-f)N$ of sites in the limit $N\to\infty$. In particular, this analysis suggests that, in $d=2$ and $N\to\infty$,

$$\frac{t_p}{t_c}=\begin{cases}0 & \text{for } 0\leqslant f<1,\\ 1 & \text{for } f=1,\end{cases}$$

$$\frac{t_r}{t_c}=\begin{cases}0 & \text{for } f=0,\\ 1 & \text{for } 0<f\leqslant 1.\end{cases} \tag{1.5}$$

Therefore the cost of time t_p to visit a fraction $f<1$ of unselected sites is negligible compared to the cost of time t_c to visit all sites, whereas the cost of time t_r to visit a fraction $f>0$ of previously selected sites is comparable to the cost of time t_c to visit all sites. In fact, for large N and f fixed, the MC data [8] suggest

$$\frac{t_p}{t_c}\approx-\frac{\ln(1-f)}{\ln N}, \tag{1.6}$$

a behavior in agreement with analytic results [7,9] for $t_p/t_c \geqslant 1/\ln N$. On the other hand, for t_r/t_c the situation is quite complex [10]. MC studies [8] suggest

$$\frac{t_r}{t_c} \approx 1 + \frac{\ln f^*}{\ln N^*}, \tag{1.7}$$

where for $f = m/N \to 0$, one fixes $f^* = ef$, where $\ln e = 1$, and $N^* = N$ by using the asymptotic expression for t_c in Eq. (1.2) and matching Eq. (1.7) to Montroll's result for the ''one-trap'' problem [11]. For $0 \leqslant f \leqslant 1$ a good fitting of the MC data is obtained for $f^* = f$ and $N^* = N^{1.2}$.

In this work we consider the PCT and RCT problems in one dimension. We start by presenting a generalized first-passage time problem on a d-dimensional lattice. Of particular interest is the investigation of the role played by dimensionality in these problems. Indeed, it is precisely the simplifying feature introduced in the formalism for the one-dimensional case that allows its exact solution. Moreover, it is shown that our derived results, complemented by MC simulations, exhibit quite distinct features in comparison with the behavior discussed above for the two-dimensional case.

II. GENERALIZED FIRST-PASSAGE TIME PROBLEM

We will consider random walks on a finite lattice with N sites and an arbitrary transition probability $p(s|s')$ of stepping from site s to the site s'. Let $P_t(s|s_0)$ denote the site occupation probability, that is, the probability that the walker is at site s after t steps, given that the walker started at site s_0. The site occupation generation function, also called the lattice Green function,

$$P(z;s|s_0) = \sum_{t=0}^{\infty} P_t(s|s_0) z^t, \tag{2.1}$$

plays a central role in the study of lattice random walks [11–13,2]. The first-passage time probability $F_t(s|s_0)$ is the probability of arriving at site s for the first time on the tth step, given that the walker started at site s_0. The first-passage time generating function is defined by

$$F(z;s|s_0) = \sum_{t=1}^{\infty} F_t(s|s_0) z^t. \tag{2.2}$$

The generating function $F(z;s|s_0)$ can be expressed in terms of $P(z;s|s_0)$ using a basic reasoning [12] which we will employ twice again later in this work. The event ''the walker is at site s after t steps'' can be decomposed into t disjoint events ''the walker first arrived at site s after t' steps, and subsequently performed $t-t'$ steps returning to the site s.'' Thus,

$$P_t(s|s_0) = \delta_{t,0}\delta_{s,s_0} + \sum_{t'=1}^{t} F_{t'}(s|s_0) P_{t-t'}(s|s). \tag{2.3}$$

Multiplying by z^t and summing over t we find

$$F(z;s|s_0) = \frac{P(z;s|s_0) - \delta_{s,s_0}}{P(z;s|s)}. \tag{2.4}$$

Let $t(s_j|s_i)$ denote the first-passage time from site s_i to site s_j. The mean first-passage time from site s_i to site s_j is given by

$$t_{ij} = \langle t(s_j|s_i) \rangle = \sum_{t=1}^{\infty} t F_t(s_j|s_i) = \frac{\partial}{\partial z} F(z;s_j|s_i) \bigg|_{z=1}. \tag{2.5}$$

We will assume that the random walk is directionally unbiased, that is, $p(s'|s) = p(s|s')$. Then,

$$P(z;s_j|s_i) = P(z;s_i|s_j), \quad F(z;s_j|s_i) = F(z;s_i|s_j), \quad t_{ij} = t_{ji}. \tag{2.6}$$

Explicit results for the mean first-passage time, Eq. (2.5), for d-dimensional translationally invariant lattices can be found in Ref. [13].

We now consider the first-passage time in relation to a given set of m distinct sites $s_1, s_2, \dots, s_m$. Let us begin by introducing the conditional first-passage time probability $F_t^\dagger(s_i|s_0)$ of arriving at site s_i for the first time on the tth step, given that the walk started at site s_0 and avoided the sites $s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_m$. Here $F_t^\dagger(s_i|s_0)$ can be thought of as the first-passage time probability in the presence of absorbing defects or traps at the sites $s_1, \dots, s_m$. Since the event ''the walker is at site s_i after t steps, given that the walk started at site s_0'' can be decomposed into tm disjoint events ''the walker arrives at site s_j for the first time on the t'th step, given that the walk started from site s_0 and avoided the sites $s_1, \dots, s_{j-1}, s_{j+1}, \dots, s_m$, and subsequently performed $t-t'$ steps to reach site s_i,'' we have

$$P_t(s_i|s_0) = \sum_{t'=1}^{t} \sum_{j=1}^{m} F_{t'}^\dagger(s_j|s_0) P_{t-t'}(s_i|s_j). \tag{2.7}$$

Introducing the conditional first-passage time generating function

$$F^\dagger(z;s|s_0) = \sum_{t=1}^{\infty} F_t^\dagger(s|s_0) z^t, \tag{2.8}$$

we obtain

$$P(z;s_i|s_0) = \sum_{j=1}^{m} F^\dagger(z;s_j|s_0) P(z;s_i|s_j). \tag{2.9}$$

Using Eq. (2.4) we may relate the conditional first-passage generating function to the first-passage generating function:

$$F(z;s_i|s_0) = F^\dagger(z;s_i|s_0) + \sum_{j \neq i} F^\dagger(z;s_j|s_0) F(z;s_i|s_j). \tag{2.10}$$

By letting $i = 1, 2, \dots, m$ in the above equation we obtain a system of m linear equations for $F^\dagger(z;s_i|s_0)$ which can be solved in terms of the first-passage time generating function. For $m = 1$ we have, trivially,

$$F^\dagger(z;s_1|s_0) = F(z;s_1|s_0) \tag{2.11}$$

and, for $m=2$,

$$F^{\dagger}(z;s_1|s_0)=\frac{F(z;s_1|s_0)-F(z;s_2|s_0)F(z;s_1|s_2)}{1-F(z;s_1|s_2)^2},$$

$$F^{\dagger}(z;s_2|s_0)=\frac{F(z;s_2|s_0)-F(z;s_1|s_0)F(z;s_2|s_1)}{1-F(z;s_1|s_2)^2}. \quad (2.12)$$

The probability that the site s_j is visited by the walker without going through sites $s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_m$, given that the walk started at site s_0, is

$$f_j^{\dagger}=\sum_{t=1}^{\infty} F_t^{\dagger}(s_j|s_0)=F^{\dagger}(1;s_j|s_0). \quad (2.13)$$

Let $t^{\dagger}(s_j|s_i)$ denote the conditional first-passage time from site s_i to site s_j. The mean conditional first-passage time from site s_i to site s_j is given by

$$t_{ij}^{\dagger}=\langle t^{\dagger}(s_j|s_i)\rangle=\sum_{t=1}^{\infty} tF_t^{\dagger}(s_j|s_i)=\frac{\partial}{\partial z}F^{\dagger}(z;s_j|s_i)\bigg|_{z=1}. \quad (2.14)$$

The mean time to reach any of m sites $s_1, \ldots, s_m$ for the first time starting from site s_0 is

$$t^{\dagger}=\sum_{j=1}^{m} t_{0j}^{\dagger}=\sum_{j=1}^{m} \frac{\partial}{\partial z}F^{\dagger}(z;s_j|s_0)\bigg|_{z=1}. \quad (2.15)$$

We next define the generalized first-passage time probability $F_t(s_1,s_2, \ldots, s_m|s_0)$ that the walker visits all sites $s_1,s_2, \ldots, s_m$ for the first time on the tth step, given that the walk started at site s_0. The event ''the walker visited all sites $s_1,s_2, \ldots, s_m$ for the first time on the tth step, given that the walk started at site s_0'' can be decomposed into tm disjoint events ''the walker arrived at site s_i for the first time on the t'th step, given that the walk started at site s_0 and avoided the sites $s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_m$, and subsequently visited these sites for the first time on the $(t-t')$th step.'' Thus,

$$F_t(s_1,s_2, \ldots, s_m|s_0)=\sum_{t'=1}^{t}\sum_{i=1}^{m} F_{t'}^{\dagger}(s_i|s_0)\times F_{t-t'}\times(s_1,s_2, \ldots, s_{i-1},s_{i+1}, \ldots, s_m|s_i). \quad (2.16)$$

Introducing the generalized first-passage generating function

$$F(z;s_1,s_2, \ldots, s_m|s_0)=\sum_{t=1}^{\infty} F_t(s_1,s_2, \ldots, s_m|s_0)z^t, \quad (2.17)$$

we obtain

$$F(z;s_1,s_2, \ldots, s_m|s_0)=\sum_{i=1}^{m} F^{\dagger}(z;s_i|s_0)F(z;s_1, \ldots, s_{i-1},s_{i+1}, \ldots, s_m|s_i). \quad (2.18)$$

Using Eq. (2.12) we find, for $m=2$,

$$F(z;s_1,s_2|s_0)=\left[\frac{F(z;s_1|s_0)+F(z;s_2|s_0)}{1+F(z;s_1|s_2)}\right]F(z;s_1|s_2). \quad (2.19)$$

In principle it is possible to determine the generating functions $F(z;s_1,s_2, \ldots, s_m|s_0)$ recursively using Eqs. (2.10) and (2.18), but the resulting expressions become very lengthy already for $m=3$.

Let $t(s_1, \ldots, s_m|s_0)$ denote the generalized first-passage time through sites $s_1,s_2, \ldots, s_m$ starting from site s_0. The mean generalized first-passage time is given by

$$\langle t(s_1, \ldots, s_m|s_0)\rangle=\sum_{t=1}^{\infty} tF_t(s_1, \ldots, s_m|s_0)=\frac{\partial}{\partial z}F(z;s_1, \ldots, s_m|s_0)\bigg|_{z=1}. \quad (2.20)$$

Differentiating Eq. (2.10) with respect to z and setting $z=1$, we find

$$t^{\dagger}=t_{0i}-\sum_{j=1}^{m} t_{ij}f_j^{\dagger} \quad (i=1,2, \ldots, m), \quad (2.21)$$

where $f_j^{\dagger}$ and $t^{\dagger}$ have been defined in Eqs. (2.13) and (2.14), respectively, and we have adopted the convention $t_{ii}=0$. The probabilities $f_j^{\dagger}$ satisfy the system of m linear equations

$$\sum_{j=1}^{m} f_j^{\dagger}=1, \quad (2.22)$$

$$\sum_{j=1}^{m} (t_{1j}-t_{ij})f_j^{\dagger}=t_{01}-t_{0i} \quad (i=2,3, \ldots, m). \quad (2.23)$$

Equation (2.22) follows from the fact that $F_t^{\dagger}(s_i|s_0)$ is a joint probability distribution in t and s_i, whereas Eq. (2.23) results from Eq. (2.21). Differentiating Eq. (2.18) with respect to z and setting $z=1$, we find

$$\langle t(s_1, \ldots, s_m|s_0)\rangle=t^{\dagger}+\sum_{j=1}^{m} f_j^{\dagger}\langle t(s_1, \ldots, s_{j-1},s_{j+1}, \ldots, s_m|s_j)\rangle. \quad (2.24)$$

Using the result (2.21) for $t^{\dagger}$ we may write

$$\langle t(s_1,\ldots,s_m|s_0)\rangle = t_{0i}-\sum_{j=1}^{m} t_{ij}f_j^{\dagger}+\sum_{j=1}^{m} f_j^{\dagger} \times\langle t(s_1,\ldots,s_{j-1},s_{j+1},\ldots,s_m|s_j)\rangle, \quad (2.25)$$

where $f_j^{\dagger}$ are determined from the system of equations (2.22) and (2.23). Thus we have a recursion relation for the generalized mean first-passage time for *m* sites in terms of $m-1$ sites. For $m=2$ we find the simple result

$$\langle t(s_1,s_2|s_0)\rangle = \frac{1}{2}(t_{01}+t_{12}+t_{20}), \quad (2.26)$$

but already for $m=3$ the expression becomes rather lengthy:

$$\begin{aligned}\langle t(s_1,s_2,s_3|s_0)\rangle = (&3t_{12}^3+3t_{13}^3+3t_{23}^3+2t_{10}t_{23}^2+2t_{12}^2t_{30}\\ &+2t_{13}^2t_{20}-3t_{12}^2t_{13}-3t_{12}^2t_{23}-3t_{12}t_{13}^2\\ &-3t_{12}t_{23}^2-3t_{13}^2t_{23}-3t_{13}t_{23}^2-2t_{10}t_{12}t_{23}\\ &-2t_{10}t_{13}t_{23}-2t_{12}t_{13}t_{20}-2t_{12}t_{13}t_{30}\\ &-2t_{12}t_{23}t_{30}-2t_{13}t_{20}t_{23}-14t_{12}t_{13}t_{23})/2\Delta,\end{aligned} \quad (2.27)$$

where

$$\Delta = 2t_{12}t_{13}+2t_{12}t_{23}+2t_{13}t_{23}-t_{12}^2-t_{13}^2-t_{23}^2. \quad (2.28)$$

Thus, although the generalized mean first-passage time can be written in terms of the first-passage time for arbitrary lattices, it does not seem to be practical in the situation where *m* is large. However, the topological constraint proper to one-dimensional lattices simplifies the formalism drastically and allows for a full solution of the problem.

III. ONE-DIMENSIONAL CASE

In this section we obtain explicit results for the generalized first-passage time problem in one-dimensional lattices with periodic boundary conditions. We consider a ring of *N* sites with coordinates $s=0,1,2,\ldots,N-1$. A walker steps between nearest-neighbor sites with equal probability 1/2. Without loss of generality we will assume that $s_0<s_1<\cdots<s_m$. Starting from s_0 it is impossible to reach the sites $s_2,\ldots,s_{m-1}$ without passing through s_1 or s_m. Thus $f_j^{\dagger}=0$ for $j=2,\ldots,m-1$, and the system (2.22) and (2.23) becomes

$$f_1^{\dagger}+f_m^{\dagger}=1, \quad (3.1)$$

$$-t_{1m}f_1^{\dagger}+t_{1m}f_m^{\dagger}=t_{01}-t_{0m}, \quad (3.2)$$

where we have chosen $i=m$ in Eq. (2.23). Solving for $f_1^{\dagger}$ and $f_m^{\dagger}$ and inserting into Eq. (2.25) we find

$$\langle t(s_1,\ldots,s_m|s_0)\rangle = \frac{1}{2}(t_{01}+t_{0m}-t_{1m}) +f_1^{\dagger}\langle t(s_2,\ldots,s_m|s_1)\rangle +f_m^{\dagger}\langle t(s_1,\ldots,s_{m-1}|s_m)\rangle. \quad (3.3)$$

For $m=2$ we recover the result given by Eq. (2.26). Let us prove that the mean generalized first-passage time is given by *half the sum of the mean first-passage time between neighboring sites*. The assumption is clearly valid for $m=2$. Assuming the validity for a visit to $m-1$ sites we have

$$\langle t(s_2,\ldots,s_m|s_1)\rangle = \langle t(s_1,\ldots,s_{m-1}|s_m)\rangle = \frac{1}{2}\left(\sum_{i=1}^{m-1} t_{i,i+1}+t_{m1}\right). \quad (3.4)$$

Substitution into Eq. (3.3) gives

$$\langle t(s_1,\ldots,s_m|s_0)\rangle = \frac{1}{2}\left(\sum_{i=0}^{m-1} t_{i,i+1}+t_{m0}\right), \quad (3.5)$$

which shows that the assertion is valid for *m*. Thus we have proven by induction the validity of our assertion for all $m\geqslant 2$.

The mean first-passage time for a one-dimensional lattice with periodic boundary conditions and *N* sites [13] is given by

$$t_{ij}=(s_j-s_i)(N-s_j+s_i) \quad (s_j>s_i). \quad (3.6)$$

Let us introduce the variables n_i defined by

$$n_i=s_{i+1}-s_i-1 \quad (i=0,1,\ldots,m-1),$$
$$n_m=N+s_0-s_m-1. \quad (3.7)$$

n_i counts the number of lattice sites between s_i and s_{i+1} and satisfies

$$n_i\geqslant 0, \quad \sum_{i=0}^{m} n_i=N-m-1. \quad (3.8)$$

In terms of these counting variables Eq. (3.5) for the mean first-passage time can be written

$$\langle t(s_1,\ldots,s_m|s_0)\rangle = \frac{1}{2}\left[(N-1)^2+m-\sum_{i=0}^{m} n_i^2\right]. \quad (3.9)$$

IV. COVERING TIMES IN ONE DIMENSION

In a lattice with *N* sites there are at most $N-1$ sites that can be visited for the first time from a given starting site. The CT problem corresponds to $m=N-1$, and can be obtained from Eq. (3.9) by setting $m=N-1$ and $n_i=0$ for all *i*. The result is

$$t_c=\langle t(1,\ldots,N-1|0)\rangle = \frac{1}{2}N(N-1), \quad (4.1)$$

in agreement with previous studies [3,6].

In the PCT problem, we are interested in the expected number of steps t_p taken by the walker to visit m sites for the first time. In a one-dimensional lattice with periodic boundary conditions the PCT problem is equivalent to a CT problem of a lattice with $m+1$ sites. Therefore,

$$t_p=\frac{1}{2}m(m+1). \tag{4.2}$$

In the RCT problem the m sites $s_1, s_2, \ldots, s_m$ are chosen at random. There is a one-to-one correspondence between the set of numbers s_i and n_i given by Eq. (3.7). Thus the expected time taken to visit all m sites for the first time is

$$t_r=\langle\langle t(s_1,\ldots,s_m|s_0)\rangle\rangle=\frac{1}{2}\left[(N-1)^2+m-\sum_{i=0}^{m}\langle n_i^2\rangle\right], \tag{4.3}$$

where $\langle\cdots\rangle$ is the mean with respect to the distribution of n_i. Due to the condition (3.8), we recognize n_i to be the occupation number of the ith state of $N-m-1$ particles obeying Bose-Einstein statistics, i.e., the statistics of distributing $q=N-m-1$ indistinguishable balls among $r=m+1$ urns. The probability $p_{q,r}(n)$ of finding n balls in an urn is given by [14]

$$p_{q,r}(n)=\binom{q+r-n-2}{q-n}\Bigg/\binom{q+r-1}{q-1}, \tag{4.4}$$

with $p_{q,r}(n)=0$ for $n<0$ and $n>q$. To compute the kth moments

$$\langle n^k\rangle_{q,r}=\sum_{n=0}^{\infty} n^k p_{q,r}(k), \tag{4.5}$$

it is convenient to observe that $p_{q,r}(n)$ obeys the recursion relation

$$p_{q,r}(n)-p_{q,r}(n-1)+\left(\frac{r-1}{q+r-1}\right)p_{q,r-1}(n-1)=0. \tag{4.6}$$

Multiplying by $(n-1)^k$ and summing over n we find

$$\sum_{l=0}^{k-1}\binom{k}{l}(-1)^{k-l}\langle n^l\rangle_{q,r}+\left(\frac{r-1}{q+r-1}\right)[\langle n^k\rangle_{q,r-1}-(-1)^k]=0, \tag{4.7}$$

which permits the computation of $\langle n^k\rangle_{q,r}$ recursively. For $k=1$ and $k=2$ we find

$$\langle n\rangle_{q,r}=\frac{q}{r}=\frac{N-m-1}{m+1}, \tag{4.8}$$

$$\langle n^2\rangle_{q,r}=\frac{q(2q+r-1)}{r(r+1)}=\frac{(N-m-1)(2N-m-2)}{(m+1)(m+2)}. \tag{4.9}$$

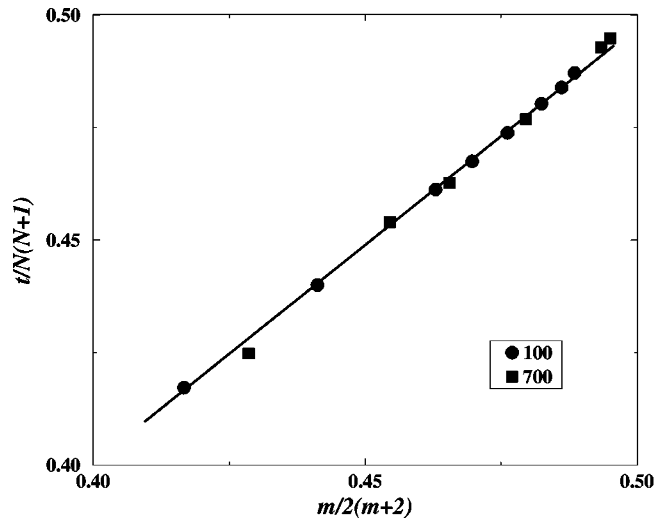

FIG. 1. Results of MC simulations for RCT in a ring with $N=100$ and 700 sites, with $10\leqslant m\leqslant 200$ random sites to be visited. The scaling plot shows $t/N(N+1)$ as a function of $m/2(m+2)$. The data collapsed on the straight line with slope 0.98 ± 0.03.

Finally, substitution of the result (4.9), which is the same for all $m+1$ urns, into Eq. (4.3) gives

$$t_r=\frac{m}{2(m+2)}N(N+1). \tag{4.10}$$

For $m=1$ this result coincides with the mean times required for the walker to be trapped by an absorbing site [11], whereas for $m=N-1$ we recover the covering time t_c given by Eq. (4.1). The results of the MC simulations shown in Fig. 1 further confirm the analytic result (4.10) for the RCT.

Let $f=m/N$ denote the fraction of lattice sites to be visited. Then, from Eqs. (4.1), (4.2), and (4.10) we find

$$\frac{t_p}{t_c}=f^2+\frac{1-f}{N+1},\quad \frac{t_r}{t_c}=1-\frac{2}{fN+2}. \tag{4.11}$$

The results of MC simulations shown in Fig. 2 are in good agreement with the above equations. Thus for $d=1$ and in the limit $N\to\infty$ we have

$$\frac{t_p}{t_c}=f^2,\quad \frac{t_r}{t_c}=\begin{cases}\dfrac{m}{m+2} & \text{for } f=0,\\ 1 & \text{for } 0<f\leqslant 1.\end{cases} \tag{4.12}$$

Therefore the cost of time t_p to visit a fraction f of unselected sites is proportional to the cost of time t_c to visit all sites and increases as f^2, whereas the cost of time t_r to visit a fraction $f=m/N\to 0$ of previously selected sites is proportional to t_c and increases as $m/(m+2)$.

V. CONCLUSIONS

In this work we have solved the PCT and RCT problems exactly in one dimension. We first generalized the notion of first-passage time to a set of m sites for arbitrary lattices. Its generating function can be expressed in terms of a first-

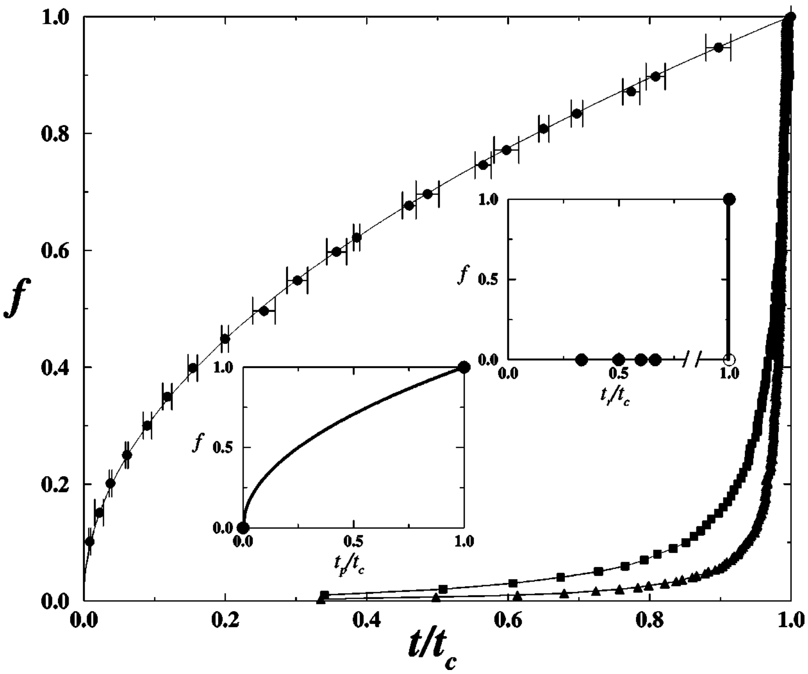

FIG. 2. Results of MC simulations for RCT in a ring with $N=100$ (squares), 300 (triangles), and 1000 (circles) sites. The graph shows the fraction f of sites visited as a function of scaled time t/t_c. The points to the left correspond to the PCT, the solid curve being the best fit given by $(t/t_c)^{0.486\pm 0.002}$. The points to the right correspond to the RCT. The inset shows the result for the infinite lattice limit $N\rightarrow\infty$.

passage generating function. As a consequence, the mean generalized first-passage time can be written in terms of mean first-passage times, although the resulting expression becomes very complicated already for $m=3$ sites. However, in the one-dimensional case a drastic simplification occurs, which allowed us to obtain a general expression for the mean generalized first-passage time. This result was then used to compute the covering time t_c, partial covering time t_p, and random covering time t_r. In the limit of infinite lattice size, the behavior of the ratios t_p/t_c and t_r/t_c as a function of the fraction of sites f is shown to be quite distinct in one and two dimensions. In fact, in one dimension the cost of time t_p to visit a fraction $f>0$ of unselected sites is proportional to the cost of time t_c to visit all N sites and increases as f^2, whereas in two dimensions this cost is negligible except for $f=1$. Our result for t_r generalizes the ''one-trap'' problem and shows that in one dimension the cost to visit a fraction $f=m/N\rightarrow 0$ of previously selected sites is proportional to t_c and increases as $m/(m+2)$, thus being sensitive to the number m of trap sites, whereas for $0<f\leqslant 1$, $t_r=t_c$. In particular, this result clarifies, for the one-dimensional case, the suggestion of Weiss *et al.* in Ref. [7] that ''the single trap approximation is indeed a valid low-concentration limit for survival on an infinite lattice with a finite concentration of traps.'' It changes drastically in two dimensions, in which case the cost t_r to visit a fraction $f\rightarrow 0$ of previously selected sites is negligible in comparison with the cost t_c to visit all N sites. Although it is clear that in two dimensions there are much more possibilities for the walker to wander around the lattice, we believe that some of the results derived in this work, valid in one dimension, could not be easily anticipated without a rigorous analysis of the problem.

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